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# PHASE-INDEPENDENT METHODS FOR THE INVERSE DIFFRACTION PROBLEM

NOVEMBER 1967

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R. M. Lewis (Consultant)

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Prepared for

DEPUTY FOR SURVEILLANCE AND CONTROL SYSTEMS
SPACE DEFENSE SYSTEMS PROGRAM OFFICE
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts



Project 4966
Prepared by
THE MITRE CORPORATION
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## FOREWORD

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ROBERT L. EDGE, Colonel, USAF
Director, Space Defense Systems Program Office
Deputy for Surveillance and Control Systems

## ABSTRACT

An earlier method for the inverse diffraction problem, using only the backscattering cross section as a function of aspect angle, yields the profile curve for a strictly convex body of revolution. If the profile curve has straight line segments, the body is not strictly convex. Nevertheless, it is shown that the method, when applied to such a body, yields the correct profile except for a shortening of the straight segment by a factor which is at worst 0.7.

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## SECTION I

## INTRODUCTION

One of the simplest approaches to the inverse diffraction problem is based on the geometrical optics backscattering cross section, $\sigma$. For a smooth strictly convex body of revolution, $\sigma$ is simply related to the curvature $\kappa$ of the profile curve. In References [1]* and [2] it was shown that from a knowledge of $\sigma$ as a function of aspect angle one could recover the profile curve by integrating a simple ordinary differential equation. The cross section function $\sigma$ is discussed in Section II, and, in Section III, we present a simplified version of the earlier theory, leading to an integral formula for the profile function.

A body of revolution whose profile curve contains a straight line segment (such as a cone or a cylinder) is not strictly convex. Nevertheless, it was found experimentally in [3] that, in certain cases, satisfactory results could be obtained for such targets using the method derived in Section III. In order to understand the domain of applicability of the method we therefore apply it, in Section IV, to the case of an arbitrary straight line segment. This corresponds to a conical section of the target. In so doing we use the physical optics representation for $\sigma$. This leads to a triple integral. Two of the integrations can be carried out approximately by the two-dimensional stationary phase method, and the remaining integration can be carried out exactly. We find that the resulting profile reproduces the assumed straight line segment except for an error in the axial component of the segment. The error vanishes for a cylindrical section and is a maximum for a full

*Numbers in brackets denote references cited on page 17.

(truncated) cone. In that case we find that the cone is shortened by a factor of approximately 0.7, a result which agrees almost perfectly with experimental results reported in [ 3 ] .

## SECTION II

## PHYSICAL OPTICS AND GEOMETRICAL OPTICS REPRESENTATIONS FOR THE BACKSCATTERING CROSS SECTION

We consider a perfectly conducting body of revolution illuminated by a plane wave (see Figure 1)

$$\underline{E}_i = \underline{E}_0 \, e^{ik\underline{I}\underline{X}} \tag{1}$$

where $\underline{E}_0$ is a constant vector, $k = \omega/c$ is the wave number, and

$$\underline{I} = -(\cos \theta, \ \sin \theta) \tag{2}$$

is a unit vector in the direction of incidence. The plane of the figure is chosen to contain the vector $\underline{I}$ and the z-axis, which is the axis of rotation.

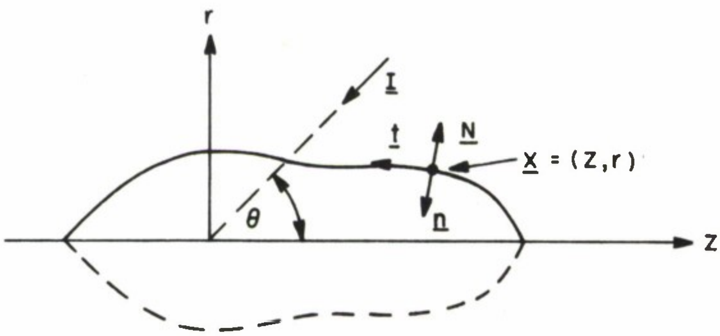


Figure 1. Perfectly Conducting Body of Revolution Illuminated by a Plane Wave

By means of physical optics (the Kirchhoff method), the scattered field can be represented as a surface integral over the illuminated portion of the target. This integral can then be reduced to a single integral over the profile curve (solid line in Figure 1) by the stationary phase method. The details of this calculation are given in [ 4 ] . The backscattered far-field at the point $\underline{X}' = -\rho \underline{I}$ is then given by (see Equations (2.2) and (2.13) of [4] ):

$$\underline{E}_s \sim \frac{-ike^{ik\rho}}{2\pi\rho} f \underline{E}_0 \tag{3}$$

where

$$f = \left[\frac{\pi}{k\sin\theta}\right]^{1/2} e^{i\pi/4} \int r^{1/2} \underline{I}\cdot\underline{N} e^{2ik\underline{I}\cdot\underline{X}} ds \quad . \tag{4}$$

Here $\underline{X} = \underline{X}(s) = [z(s), r(s)]$ is the parametric equation of the profile curve, $\underline{N} = -\underline{n}$ is the outward unit vector normal to the curve, and $s$ is the arclength parameter. For simplicity we have assumed that only the upper part of the target (solid line in Figure 1) is illuminated.

We introduce the function

$$g(\theta) = \int r^{1/2} \underline{I}\cdot\underline{n} \; e^{2ik\underline{I}\cdot\underline{X}} ds \quad . \tag{5}$$

Then

$$\underline{E}_s \sim \left[\frac{k}{\pi\sin\theta}\right]^{1/2} \frac{e^{ik\rho+3i\pi/4}}{2\rho} g \underline{E}_0 \quad . \tag{6}$$

We also introduce the backscattering cross section

$$\sigma = 4\pi\rho^2 \frac{|E_S|^2}{|E_0|^2} = \frac{k}{\sin\theta}|g|^2 . \tag{7}$$

If the profile curve is strictly convex, the integral in Equation (5) can be evaluated by the stationary phase method, which is summarized in the Appendix. We introduce the phase function

$$\phi(s) = 2\underline{I}\cdot\underline{X}(s) \tag{8}$$

which appears in Equation (5). We also introduce the unit tangent vector $\underline{t} = \underline{X}(s) = \frac{dX}{ds}$ and the unit normal vector $\underline{n} = -\underline{N}$ to the profile curve.

Then

$$\ddot{\underline{X}} = \dot{\underline{t}} = \underline{n} \tag{9}$$

where $\kappa$ is the curvature. At a stationary point s for Equation (5),

$$\dot{\phi}(s) = 2\underline{I}\cdot\underline{t} = 0 . \tag{10}$$

This is, of course, the specular point, at which $\underline{n} = \underline{I}$. At this point,

$$\phi = 2\underline{I}\cdot\ddot{\underline{X}} = 2\kappa\underline{I}\cdot\underline{n} = 2 , \tag{11}$$

and the stationary phase formula [Equation (50), with n = 1] yields

$$g(\theta) \sim \left[\frac{\pi r(\theta)}{k(\theta)}\right]^{1/2} e^{2ik\underline{I}\cdot\underline{X}(\theta) + i\pi/4} . \tag{12}$$

Here $\underline{X}(\theta) = [\ z(\theta),\ r(\theta)]$ is the specular point and $\kappa(\theta)$ is the curvature at that point. From Equation (7) we now see that

$$\sigma = \frac{\pi r(\theta)}{\sin\theta\kappa(\theta)} . \tag{13}$$

It is interesting to note that the two principal curvatures of the surface of revolution at the specular point are $\kappa$ and $(\sin\theta)/r$. Thus the Gaussian curvature is $G = (\kappa\sin\theta)/r$ and Equation (13) becomes

$$\sigma = \frac{\pi}{G} . \tag{14}$$

This is a well-known result which can be obtained directly by geometrical optics. We have derived it by means of the physical optics representation because we will soon consider the case in which the profile curve has a straight line segment. Then $\kappa = 0$ and Equation (13) is not valid. In such cases the scattering cross section is given by Equations (7) and (5).

## SECTION III

## SOLUTION OF THE INVERSE DIFFRACTION PROBLEM FOR STRICTLY CONVEX BODIES OF REVOLUTION

At the stationary point $\underline{X}(\theta)$ we see, from Equation (10), that

$$dz \cos \theta + dr \sin \theta = 0 . \tag{15}$$

Furthermore, by definition

$$\kappa(\theta) = \frac{d\theta}{ds} , \tag{16}$$

and

$$\frac{ds}{dr} = \pm \sqrt{\frac{dz^2 + dr^2}{dr^2}} = \pm \sqrt{1 + \tan^2 \theta} = \frac{1}{\cos\theta} . \tag{17}$$

Hence, from Equation (13),

$$\dot{\sigma} = \frac{\pi r}{\sin\theta\kappa} = \frac{\pi r}{\sin\theta} \frac{ds}{dr} \frac{dr}{d\theta} = \frac{\pi r}{\sin\theta\cos\theta} \frac{dr}{d\theta} . \tag{18}$$

It follows from Equations (15) and (18) that

$$dr = \frac{\sigma}{\pi r} \sin\theta\cos\theta d\theta, \tag{19}$$

and

$$dz = - \frac{\sigma}{\pi r} \sin^2 \theta \, d\theta . \tag{20}$$

These equations can easily be integrated to yield

$$r^2(\theta) = r^2\left(\theta_0\right) + \frac{2}{\pi} \int_{\theta_0}^{\theta} \sigma(\theta')\sin\theta'\cos\theta'd\theta' \tag{21}$$

and

$$z(\theta) = z\left(\theta_0\right) - \frac{1}{\pi} \int_{\theta_0}^{\theta} \frac{\sigma(\theta')}{r(\theta')} \sin^2\theta'd\theta' \; . \tag{22}$$

Equations (21) and (22) provide an explicit parametric representation for the profile curve $\underline{X} = [\, z(\theta),\ r(\theta)]$ in terms of the scattering cross section $\sigma(\theta)$.

## SECTION IV

## APPLICATION TO A CONICAL SECTION

We now consider a body of revolution, a portion of which is a conical section. The corresponding profile curve then has a straight line segment as illustrated in Figure 2. On this segment the curvature $\kappa$ is zero and the derivation in Section III does not apply. Nevertheless, we shall examine the consequences of applying Equations (21) and (22) to this case, using for $\sigma(\theta)$ the physical optics scattering cross section. The first step is to calculate $\sigma$.

From Figure 2 we see that

$$\underline{n} = (-\cos\beta , -\sin\beta ), \ \underline{t} = (-\sin\beta , \cos\beta ) \tag{23}$$

and

$$\underline{X} = [ z(s), r(s)] = (z_0, r_0) + s(-\sin\beta , \cos\beta ), \ 0 \le s \le L \ . \tag{24}$$

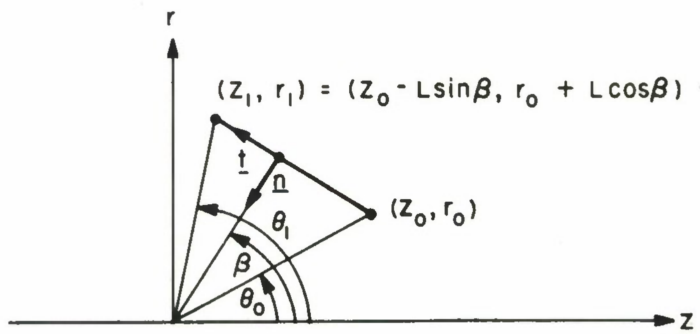


Figure 2. Profile Curve for Body of Revolution with Conical Section

The coordinates of the terminal point are then

$$\left(z_1, r_1\right) = [\, z(L),\ r(L)] = \left(z_0 - L\sin\beta\ ,\ r_0 + L\cos\beta\right) , \tag{25}$$

and

$$\underline{I}\cdot\underline{n} = \cos\theta\cos\beta + \sin\theta\sin\beta = \cos(\theta - \beta) \ . \tag{26}$$

We also see that

$$\underline{I}\cdot\underline{X} = -\ z_0\cos\theta - r_0\sin\theta - s\sin(\theta - \beta) \ . \tag{27}$$

We now assume that for $\theta_0 \leq \theta \leq \theta_1$ , contributions to the scattering cross section from portions of the target other than the conical section can be neglected. Then, from Equation (5),

$$g(\theta) = \cos(\theta - \beta)e^{-2ik\left(z_0\cos\theta + r_0\sin\theta\right)} \int_0^L \left(r_0 + s\cos\beta\right)^{1/2} e^{-2iks\sin(\theta-\beta)}ds, \tag{28}$$

and, from Equation (7),

$$\sigma(\theta) = \frac{k}{\sin\theta}\,|g|^2 = \frac{k}{\sin\theta}\cos^2(\theta-\beta) \int_0^L ds \int_0^L dt\left(r_0 + s\cos\beta\right)^{1/2}\left(r_0 + t\cos\beta\right)^{1/2} e^{2ik(t-s)\sin(\theta-\beta)} . \tag{29}$$

We now insert Equation (29) into (21) to obtain

$$r^2(\theta) - r_0^{\,2} = \frac{2k}{\pi} \int_0^L ds \int_0^L dt \int_{\theta_0}^{\theta} d\theta' \cos^2(\theta' - \beta)$$

$$\cos\theta' \left(r_0 + s\cos\beta\right)^{1/2} \left(r_0 + t\cos\beta\right)^{1/2} e^{ik\phi} \tag{30}$$

where

$$\phi = 2(t - s)\sin(\theta' - \beta) \quad . \tag{31}$$

The integral with respect to $t$ and $\theta'$ in Equation (30) can be evaluated by the stationary phase method. Thus we apply the results of the Appendix with $n = 2$. First we note that

$$\phi_t = 2\sin(\theta' - \beta), \quad \phi_{\theta'} = 2(t - s)\cos(\theta' - \beta) \quad . \tag{32}$$

Thus there exists one stationary point at

$$t = s, \quad \theta' = \beta \quad . \tag{33}$$

At this point

$$\Phi = \begin{pmatrix} \phi_{tt} & \phi_{t\theta'} \\ \phi_{\theta' t} & \phi_{\theta'\theta'} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad . \tag{34}$$

Hence det $\Phi = -4$ , and since the two eigenvalues must have opposite signs, sig $\Phi = 0$. Thus we obtain, from Equation (50), for $\theta > \beta$ ,

$$r^2(\theta) - r_0^2 \sim \cos\beta \int_0^L ds\left(r_0 + s\cos\beta\right) = 2L\cos\beta\left(r_0 + L/2\cos\beta\right). \quad (35)$$

It must be noted that the stationary point in Equation (33) lies in the domain of integration if and only if $\theta > \beta$ . If $\theta = \beta$ , the stationary point lies on the boundary and it can be shown that the asymptotic value is one-half of Equation (35). If $\theta < \beta$ , the integral is asymptotically zero. Thus

$$r^2(\theta) \sim r_0^2 \, , \qquad \theta_0 \leq \theta < \beta \, , \quad (36)$$

$$r^2(\beta) \sim r_0^2 + L\cos\beta\left(r_0 + \frac{L}{2}\cos\beta\right) = \left(r_0 + \frac{L}{2}\cos\beta\right)^2 + \left(\frac{L}{2}\cos\beta\right)^2 , \quad (37)$$

$$r^2(\theta) \sim \left(r_0 + L\cos\beta\right)^2 , \qquad \beta < \theta \leq \theta_1 \quad . \quad (38)$$

Having determined $r(\theta)$, we now insert Equation (29) into (22). Thus

$$z(\theta) - z_0 = -\frac{k}{\pi} \int_0^L ds \int_0^L dt \int_{\theta_0}^{\theta} d\theta' \frac{1}{r(\theta')} \cos^2(\theta' - \beta)$$

$$\sin\theta' \left(r_0 + s\cos\beta\right)^{1/2} \left(r_0 + t\cos\beta\right)^{1/2} e^{ik\phi} , \quad (39)$$

where $\phi$ is again given by Equation (31). Proceeding exactly as before we obtain, for $\theta > \beta$,

$$z(\theta) - z_0 \sim \frac{\sin\beta}{r(\beta)} \int_0^L ds\,(r_0 + s\cos\beta) = - \frac{\sin\beta}{r(\beta)}\, L\, [\, r_0 + \frac{L}{2}\cos\beta] \,. \tag{40}$$

If we insert Equation (37) into (40), the result is

$$z \sim z_0 - L\sin\beta \cdot Q \tag{41}$$

where

$$Q = \left(1 + \epsilon^2\right)^{-1/2}, \qquad \epsilon = \frac{L\cos\beta}{2r_0 + L\cos\beta} \,. \tag{42}$$

Thus,

$$z(\theta) \sim z_0 \qquad \theta_0 \le \theta < \beta \,, \tag{43}$$

$$z(\theta) \sim z_0 - QL\sin\beta \qquad \beta < \theta \le \theta_1 \,; \tag{44}$$

and, from Equations (36) and (38),

$$r(\theta) \sim r_0 \,, \qquad \theta_0 \le \theta < \beta \,, \tag{45}$$

$$r(\theta) \sim r_0 + L\cos\beta \,, \qquad \beta < \theta \le \theta_1 \,. \tag{46}$$

From Figure 2 we see that the coordinates of the terminal point of the line segment are

$$\left(z_1, r_1\right) = \left(z_0 - L\sin\beta, r_0 + L\cos\beta\right) . \tag{47}$$

Thus we see that, as $\theta$ passes through the specular direction, $\theta = \beta$, the vector $\underline{X}(\theta) = [z(\theta), r(\theta)]$ jumps from the initial point to a point $[z_0 - QL\sin\beta, r_0 + L\cos\beta]$ which would agree exactly with the terminal point if Q is equal to one. Thus the term $\epsilon$ in Equation (42) represents an error which vanishes when $\beta = \pi/2$; i.e., when the conical segment reduces to a cylinder. In any case we see that

$$1 \geq Q \geq \frac{1}{\sqrt{2}} \approx 0.7 . \tag{48}$$

The method we have used to evaluate the integrals asymptotically is of the simplest kind, and does not yield the smooth transition of the profile segment from the initial to the terminal point. This transition could be obtained by means of an asymptotic expansion which is uniformly valid for $\theta$ in a neighborhood of $\theta = \beta$. Many types of integrals have been expanded uniformly, but unfortunately the integrals considered here do not reduce to one of the standard types. The uniform expansion of such integrals is under investigation.

It is interesting to note that for the case of a finite cone (see Figure 2) $r_0 = 0$ and $\epsilon = 1$. Then $Q = \frac{1}{\sqrt{2}} \approx 0.7$. In this case (in which the error is a maximum) our method yields a cone which is shortened by the factor 0.7. This result agrees almost exactly with experimental results for a cone reported in Reference [3]. For a cylinder, $\beta = \pi/2$, $\epsilon = 0$, and there is no error.

## APPENDIX

## THE METHOD OF STATIONARY PHASE IN n-DIMENSIONS

The method of stationary phase, discussed in Appendix II of Reference [5], yields the leading term of the asymptotic expansion for $k \to \infty$ of a function f(k) defined by an n-dimensional integral of the form

$$f(k) = \int_D g\left(t_1, \ldots, t_n\right) e^{ik\phi(t_1, \ldots, t_n)} dt_1 \text{---} dt_n = \int_D g(T) e^{ik\phi(T)} dT. \tag{49}$$

A stationary point $T = S = \left(s_1, \ldots, s_n\right)$ is a point for which $\frac{\partial \phi}{\partial t_\nu}(S) = 0$ for $\nu = 1, \ldots, n$. Each stationary point S in the interior of the region D yields a contribution to the leading term of the asymptotic expansion, given by

$$f(k) \sim \left(\frac{2\pi}{k}\right)^{n/2} \left[\left|\det(\phi_{\nu j})\right|\right]^{-1/2} g(S)\, e^{ik\phi(S) + \frac{i\pi}{4} \operatorname{sig}\left(\phi_{\nu j}\right)}, \quad k \to \infty. \tag{50}$$

Here $(\phi_{\nu j})$ is the matrix with elements

$$\phi_{\nu j} = \frac{\partial^2 \phi(S)}{\partial t_\nu \partial t_j}, \tag{51}$$

and sig $(\phi_{\nu j})$ denotes the "signature" of the matrix, defined by

$$\operatorname{sig}\left(\phi_{\nu j}\right) = \sum_{k=1}^{m} \operatorname{sgn} r_k . \tag{52}$$

In Equation (52), $r_1, \ldots, r_m$ are the eigenvalues of $\left(\phi_{\nu j}\right)$ and $\operatorname{sgn} r_k = \pm 1$. We assume that the determinant $\det \left(\phi_{\nu j}\right)$ is non-zero. The leading term of the asymptotic expansion of Equation (49) is obtained by summing Equation (50) over the stationary points S in D.

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13. ABSTRACT

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